The inclusion of the Schur algebra in $B(\ell^2)$ is not inverse-closed.

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Abstract

The Schur algebra is the algebra of operators which are bounded on ℓ^1 and on ℓ^∞. In [ShS], Sun conjectured that the Schur algebra is inverse-closed. In this note, we disprove this conjecture. Precisely, we exhibit an operator in the Schur algebra, invertible in ℓ^2, whose inverse is not bounded on ℓ^1 nor on ℓ^∞.

The Schur algebra is the unital algebra of infinite matrices whose rows and columns are uniformly bounded in ℓ^1. Such matrices define operators which are uniformly bounded on ℓ^p for all $1 \leq p \leq \infty$. In this short note, we prove the following

Theorem. *There exists an infinite symmetric matrix $M = \{m_{i,j}\}_{i,j\in\mathbf{N}}$ such that*

- $m_{ij} = 0$ *or* $1/4$,
- *the support of each row and each column has cardinality* 4,
- $I - M$ *is invertible in* ℓ^2, *but not in* ℓ^∞.

Proof: Let us consider a finitely generated group G, equipped with a probability measure μ on G such that $\mu(g) = \mu(g^{-1})$ for all $g \in G$, and such that the support of μ is finite and generates the group G. Let M be the operator of convolution by μ on $\ell^2(G)$, i.e.

$$M(f)(g) = \mu * f(g) = \sum_{h\in G} m(g^{-1}h)f(h) = \sum_{h\in G} m(h)f(gh).$$

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Up to enumerate the elements of G, one can see M as an infinite matrix. Note that the cardinality of the support of both the rows and the columns of M is simply the cardinality of the support of μ.

Let us check that if G is infinite, then the convolution by μ is never invertible in ℓ^∞. Note that since it is self-adjoint, it is neither invertible on ℓ^1. Let S be the support of μ. The *word metric* on G is defined as follows: $d_S(g,h) = \inf\{n \in \mathbf{N}; g^{-1}h = s_1 \dots s_n, s_i \in S\}$. The ball $B(e,n)$, of radius n and centered on the neutral element e is therefore the set of all g which can be written as a product of at most n elements of S.

For each $n \geq 1$, let f_n be the function measuring the distance to the complement of $B(e,n)$ in G, i.e.

$$f_n(g) = \min_{h \in G \setminus B(e,n)} d(g,h).$$

Obviously, f_n is a 1-Lipschitz function on (G, d_S). Therefore, by definition of M, one has that $|(I-M)(f_n)(g)| \leq 1$ for all $g \in G$. But on the other hand, $f_n(e) = n$, so we obtain the following inequality

$$\frac{\|(I-M)(f_n)\|_\infty}{\|f_n\|_\infty} \leq 1/n,$$

which tends to 0 when $n \to \infty$. Hence $I - M$ is not (left) invertible in ℓ^∞.

On the other hand, by a classical result of Kesten[1] [Kest], the group G is non-amenable if and only $I - M$ is invertible in $\ell^2(G)$. The most classical example of a non-amenable group is the free group with two generators $\langle x, y\rangle$. Taking μ such that $\mu(x) = \mu(y) = 1/4$, one gets the precise statement of the theorem. ∎

Remark 0.1. In the case of the free group with 2 generators, and for μ as above, one has $\|(I-M)^{-1}\|_2 = 2/(2-\sqrt{3})$ [Kest]. Note that this norm is just the inverse of the smallest real eigenvalue of the so-called *discrete Laplacian* $\Delta = I - M$ on the Cayley graph of the free group $\langle x, y\rangle$ (which is a 4-regular tree). Another formulation of Kesten's theorem is that a finitely generated group is non-amenable if and only if any (resp. one) of its Cayley graphs has a spetral gap (i.e. this first eigenvalue is non-zero).

References

[Cou] T. Coulhon. *Random walks and geometry on infinite graphs.* Lecture notes on analysis on metric spaces, Luigi Ambrosio, Francesco Serra Cassano, eds., 5-30, 2000.

[1] Kesten gives a probabilistic proof of his result. For a more analytic approach, one can consult for instance [Cou].

[Kest] H. Kesten. *Symmetric random walks on groups.* Trans. Amer. Math. Soc. 92, 336-354, 1959.

[ShS] C. E. Shin and Q. Sun. *Stability of localized operators.* Journal of Functional Analysis, 256, 2417–2439, 2009.